All-Multiplicity Amplitudes with Massive Scalars

Darren Forde and David A. Kosower

Service de Physique Théorique, CEA–Saclay

F–91191 Gif-sur-Yvette cedex, France

(Dated: June 3, 2005)

Abstract

We compute two infinite series of tree-level amplitudes with a massive scalar pair and an arbitrary number of gluons. We provide results for amplitudes where all gluons have identical helicity, and amplitudes with one gluon of opposite helicity. These amplitudes are useful for unitarity-based one-loop calculations in nonsupersymmetric gauge theories generally, and QCD in particular.

PACS numbers: 12.38.Bx, 12.38.-t

I. INTRODUCTION

Explicit computations of Standard Model processes will play an essential role in probing beyond it. The short-distance experimental environment at hadron colliders requires the computation of many processes involving QCD interactions. Tree-level computations do not suffice for these purposes. The coupling in QCD is sufficiently large, and varies sufficiently with scale, that even a basic quantitative understanding [1] requires the computation of next-to-leading order corrections [2] to cross sections. One-loop amplitudes are of course an essential ingredient of such corrections.

Within the unitarity-based method [3, 4, 5], one can decompose one-loop QCD amplitudes into contributions corresponding to $\mathcal{N} = 4$, $\mathcal{N} = 1$, and remaining contributions. For amplitudes where all external particles are gluons, this decomposition for the gluon-loop contributions of color-ordered amplitudes takes the following form,

$$A_n = A_n^{\mathcal{N}=4} - 4A_n^{\mathcal{N}=1} + A_n^{\text{scalar}}. \tag{1}$$

That is, the remaining contributions correspond to scalars circulating in the loop. (The contributions of quarks circulating in the loop can also be written in terms of $A^{\mathcal{N}=1}$ and A^{scalar}.) The supersymmetric contributions can be computed by performing the cut algebra strictly in four dimensions, with only the loop integrations computed in $D = 4 - 2\epsilon$ dimensions. The 'scalar' contributions require that the cut algebra, and the corresponding tree amplitudes fed into the unitarity machinery, also be computed in D dimensions [4, 6, 7, 8]. At one loop, computing a scalar loop in D dimensions is equivalent to computing a massive scalar loop, and then integrating over the mass with an appropriate weighting.

The computation of tree-level amplitudes with massive scalars is thus of use in the unitarity method for computing massless loop amplitudes in nonsupersymmetric gauge theories. The simplest such amplitudes, with up to four gluons of positive helicity, were computed by Bern, Dixon, Dunbar, and one of the authors [8]. Recently, Badger, Glover, Khoze and Svrček (BGKS) have given [9] a set of on-shell recursion relations for amplitudes with massive scalars. They used the recursion relations to compute all massive scalar amplitudes with up to four external gluons. These relations extend the tree-level on-shell recurrence relations of Britto, Cachazo, Feng, and Witten [10, 11]. On-shell recursion relations have also been applied to tree-level amplitudes by Luo and Wen [12] and by Britto, Feng, Roiban, Spradlin and Volovich [13]; to tree-level gravitational amplitudes by Bedford, Brandhuber,

Spence, and Travaglini [14], and by Cachazo and Svrček [15]; and to massive vector and quark amplitudes by Badger, Glover, and Khoze [16]. The concept has also been applied to certain loop amplitudes by Bern, Dixon, and one of the authors [17]; and to the direct calculation of some integral coefficients by Bern, Bjerrum-Bohr, Dunbar, and Ita [18]. These relations grew out of investigations [19, 20, 21, 22, 23, 24] motivated by Witten's topological twistor-string description [25] (as a weak–weak dual) of the $\mathcal{N} = 4$ supersymmetric gauge theory. The roots of this duality lie in Nair's description [26] of the simplest gauge-theory scattering amplitudes in terms of projective-space correlators. For use as building blocks in loop amplitudes, we need analytic expressions for the massive-scalar amplitudes. While the original tree-level recursive approach [27, 28, 29] pioneered by Berends and Giele is more efficient for purely numerical purposes, the on-shell approach seems better suited to providing analytic formulæ. (We discuss the computational complexity of the Berends–Giele recursion relations in the appendix.) Bern, Dixon, and one of the authors have recently presented [30] a 'unitarity-bootstrap' approach which combines four-dimensional unitarity cuts with use of a recursion relation for the rational terms. The amplitudes we compute here should be useful for proving the factorization properties in *complex* momenta required for the recursion relations part of this approach.

In this paper, we will provide formulæ for two classes of amplitudes with two color-adjacent massive scalars and n gluons, where the gluons all share the same helicity, or where one gluon has opposite helicity. In the next section, we document the notation and conventions we use; in section III, we derive a form for the all-plus amplitude, $A_n(1_s, 2^+, \ldots, (n-1)^+, n_s)$. In section IV, we derive an expression for the amplitude with a negative-helicity gluon adjacent to a scalar leg, $A_n(1_s, 2^+, \ldots, (n-1)^-, n_s)$, and extend it to one negative-helicity gluon in an arbitrary position in section V, followed by concluding remarks.

II.   NOTATION AND ON-SHELL RECURSION RELATION

We will write our expressions for various amplitudes using the spinor-helicity formalism. The formalism makes use of spinor products. We follow the conventions of the standard QCD literature, so that

$$\langle j\, l\rangle = \langle j^-|l^+\rangle = \bar{u}_-(k_j)u_+(k_l)\,, \qquad\qquad [j\, l] = \langle j^+|l^-\rangle = \bar{u}_+(k_j)u_-(k_l)\,, \qquad (2)$$

3

where $u_\pm(k)$ is a massless Weyl spinor with momentum k and positive or negative chirality [31, 32]. We take all legs to be outgoing. The two spinor products are related, with $[i\,j] = \text{sign}(k_i^0 k_j^0)\, \langle j\, i \rangle^*$ so that,

$$\langle i\, j \rangle\, [j\, i] = 2k_i \cdot k_j. \tag{3}$$

(Note that the bracket product $[i\,j]$ differs by an overall sign from that commonly used in twistor-space studies [25] and also in ref. [9].)

In the amplitudes we consider, we will also encounter sums of momenta,

$$K_{i\cdots j} = k_i + \cdots + k_j, \tag{4}$$

but in this paper the indices should *not* be interpreted in a cyclic manner; if $i > j$, $K_{i\cdots j} = 0$.

Let us also define a notation for a sequential product of spinor products,

$$\langle\!\langle j_1 \cdots j_2 \rangle\!\rangle = \langle j_1\, (j_1 + 1) \rangle\, \langle (j_1 + 1)\, (j_1 + 2) \rangle \cdots \langle (j_2 - 1)\, j_2 \rangle, \tag{5}$$

and one for the mass-subtracted square of momentum sums,

$$L_{i\cdots j} = \sum_{a=i}^{j} \sum_{b=i,\neq a}^{j} k_a \cdot k_b, \tag{6}$$

so that if $k_1^2 = m_s^2$, and $k_2, \ldots, k_j$ are massless, for example, then $L_{1\cdots j} = K_{1\cdots j}^2 - m_s^2$.

The on-shell recursion relations make use of complex momenta, obtained by shifting spinors corresponding to massless momenta. (One can also shift momenta of massive particles, but we will not need to do so in this paper.) A (j, l) shift is defined by,

$$|j^-\rangle \rightarrow |j^-\rangle - z|l^-\rangle,$$
$$|l^+\rangle \rightarrow |l^+\rangle + z|j^+\rangle, \tag{7}$$

with the remaining spinors unchanged. It gives the following shift of the momenta,

$$k_j^\mu \rightarrow \hat{k}_j \equiv k_j^\mu(z) = k_j^\mu - \frac{z}{2}\langle j^-|\,\gamma^\mu\,|l^-\rangle,$$
$$k_l^\mu \rightarrow k_l^\mu(z) = k_l^\mu + \frac{z}{2}\langle j^-|\,\gamma^\mu\,|l^-\rangle. \tag{8}$$

To obtain an explicit expression using the on-shell recursion relations, we must choose a definite shift. The relations then express an amplitude in terms of a sum over all inequivalent contiguous partitions of the external momenta into two sets, each of which contain exactly

one of the shift momenta. These partitions can thus be thought of as corresponding to sums of cyclicly-consecutive momenta. There is also a sum over the helicities of the intermediate leg. The basic relation [9, 10, 11] is,

$$A_n(k_1, \ldots, k_n) = \sum_{\text{partitions } P} \sum_{h=\pm} A_L(k_{P_1}, \ldots, \hat{k}_j, \ldots, k_{P_{-1}}, -\hat{P}^h)$$

$$\times \frac{i}{P^2 - m_P^2} A_R(k_{\overline{P}_1}, \ldots, \hat{k}_l, \ldots, k_{\overline{P}_{-1}}, \hat{P}^{-h}). \tag{9}$$

In this equation, P_1 stands for the first momentum in P, P_{-1} for the last momentum in P, and $\overline{P}$ for the complement of P (all the remaining momenta). The scalar legs (always the first and last in this paper) will be indicated by an 's' subscript. The mass of the particle in the factorized channel is m_P, and $\hat{P}$ is given by momentum conservation,

$$\hat{P} = P + \delta k_j(z) = k_{P_1} + \cdots + k_{P_{-1}} - \frac{z}{2} \langle j^- | \gamma^\mu | l^- \rangle. \tag{10}$$

In each channel, a different value of z is used here, and in eqs. (7) and (8). It is given by the on-shell condition $\hat{P}^2 - m_P^2 = 0$,

$$z = \frac{P^2 - m_P^2}{\langle j^- | \not{P} | l^- \rangle}. \tag{11}$$

As discussed in refs. [9, 11], we must ensure that $A(z)$ vanishes at large z for the recursion relations to be valid. The choices we will use have the required property, as was shown in these references.

III.   THE $A(1_s, 2^+, \ldots, (n-1)^+, n_s)$ AMPLITUDE

In this section, we provide a result for the amplitude with an arbitrary number of positive-helicity gluons. The amplitude with all negative-helicity gluons can be obtained by spinor conjugation. Let us begin at the end, by writing down the answer,

$$A_n(1_s, 2^+, \ldots, (n-1)^+, n_s) = -\frac{i}{L_{12} \langle\langle 2 \cdots (n-1) \rangle\rangle L_{(n-1)n}}$$

$$\times \sum_{j=1}^{\lfloor n/2 \rfloor - 1} (-m_s^2)^j \sum_{\substack{\{w_i\}_{i=1}^{j-1} \\ w_i = w_{i-1}+2}}^{n-3} \left(\prod_{r=1}^{j-1} \frac{\langle w_r^- | \not{K}_{(w_r+1)\cdots(w_{r+1}-1)} | w_{r+1}^- \rangle}{L_{1\cdots(w_r-1)} L_{1\cdots w_r}} \right) \langle w_1^+ | \not{K}_{2\cdots(w_1-1)} \not{k}_1 | 2^- \rangle \Bigg|_{\substack{w_0=1, \\ w_j=n-1}} \tag{12}$$

In this equation, $m_s^2 \equiv k_1^2 = k_n^2$ is the mass squared of the scalar, and $\lfloor x \rfloor$ denotes the largest integer smaller than or equal to x.

The multiple sums in eq. (12) can be written less succinctly and perhaps less forbiddingly as,

$$\sum_{\substack{\{w_i\}_{i=1}^{j-1} \\ w_i=w_{i-1}+2}}\Bigg|_{\substack{w_0=1, \\ w_j=n-1}} = \sum_{w_1=3}^{n-3}\sum_{w_2=w_1+2}^{n-3}\cdots\sum_{w_{j-1}=w_{j-2}+2}^{n-3}\Bigg|_{w_j=n-1}. \tag{13}$$

For example, in the four-point case, the product over r is absent, and we obtain,

$$\frac{im_s^2\,[3\,2]\,\langle 2^-|\,\slashed{k}_1\,|2^-\rangle}{L_{12}\,\langle 2\,3\rangle\,L_{34}} = \frac{im_s^2\,[3\,2]}{L_{12}\,\langle 2\,3\rangle}; \tag{14}$$

in the five-point case,

$$\frac{im_s^2\,\langle 4^+|\,\slashed{K}_{23}\slashed{k}_1\,|2^-\rangle}{L_{12}\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,L_{45}} \tag{15}$$

and in the six-point case, where there is only one non-trivial w variable,

$$\frac{im_s^2}{L_{12}\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,L_{56}}\left(\langle 5^+|\,\slashed{K}_{2\cdots 4}\slashed{k}_1\,|2^-\rangle - m_s^2\frac{\langle 5^+|\,\slashed{k}_4\slashed{k}_3\,|2^-\rangle}{L_{4\cdots 6}}\right) \tag{16}$$

all in agreement with refs. [8, 9] (up to an overall phase for the second reference).

Having written down the above ansatz, we now prove it using the on-shell recursion relations as given by BGKS. For this purpose, let us choose a $(3, 2)$ shift using eqs. (7) and (8). (That is, choose the 'reference' momenta to be k_3 and k_2.)

In eq. (12), the limits on the sums ensure that $i \leq j$ in sums of momenta $K_{i\cdots j}$. We can remove those limits, so long as we take $K_{i\cdots j} \equiv 0$ if $i > j$ (rather than interpreting the indices in a cyclic sense), and we shall do so. We can then rewrite eq. (12) in a form which will be more useful for the proof,

$$A_n(1_s, 2^+, \ldots, (n-1)^+, n_s) = -\frac{i}{L_{12}\,\langle\!\langle 2\cdots(n-1)\rangle\!\rangle\,L_{(n-1)n}}$$

$$\times \sum_{j=1}^{\lfloor n/2\rfloor-1}(-m_s^2)^j\sum_{\{w_i\}_{i=1}^{j-1}=3}^{n-3}\left(\prod_{r=1}^{j-1}\frac{\langle w_r^-|\,\slashed{K}_{(w_r+1)\cdots(w_{r+1}-1)}\,|w_{r+1}^-\rangle}{L_{1\cdots(w_r-1)}L_{1\cdots w_r}}\right)\langle w_1^+|\,\slashed{K}_{2\cdots(w_1-1)}\slashed{k}_1\,|2^-\rangle\Bigg|_{w_j=n-1} \tag{17}$$

We will proceed inductively, assuming that the ansatz (12) holds for the $(n-1)$-point amplitude $(n > 4)$, and showing that it holds for the n-point one. The on-shell recursion relations [9, 10, 11] tell us that the n-point amplitude can be written as a sum over all factorizations, with the two shifted momenta attached to different amplitudes in each factorization.

No factorization can isolate k_2 in a purely-gluonic amplitude. In addition, any factorization that isolates k_3 in a purely-gluonic amplitude leads to a vanishing contribution because

the amplitude vanishes. For factorizations where k_3 is isolated in a four- or higher-point amplitude, this is immediate, because all such amplitudes vanish,

$$A_{j+1}(\hat{K}^{\pm}_{3\cdots j}, 3^+, \ldots, j^+) = 0. \tag{18}$$

In the three-point case, the factorization is in the $[3\,4]$ channel, but here the relevant amplitude — $A_3(\hat{K}^{\pm}_{34}, 3^+, 4^+)$ — also vanishes.

This leaves us only with factorization in which both amplitudes have massive scalar legs, and therefore in which the factorized leg is a massive scalar. There is only one such contribution, so what we are seeking to prove is that,

$$A_n(1_s, 2^+, \ldots, (n-1)^+, n_s) = A_3(1_s, 2^+, -\hat{K}_{12s})\frac{i}{K^2_{12} - m^2_s}A_{n-1}(\hat{K}_{12s}, 3^+, \ldots, (n-1)^+, n_s). \tag{19}$$

The three-point amplitude was given by BGKS [9],

$$A_3(1_s, 2^+, 3_s) = i\frac{\langle q^-|\slashed{k}_1|2^-\rangle}{\langle q\,2\rangle}. \tag{20}$$

Its form depends on an arbitrary reference momentum q, but its value is nonetheless independent of it on shell.

Using this expression with $q = \hat{k}_3$, and rewriting denominators using momentum conservation, we have for our starting point,

$$-\frac{i}{L_{12}L_{123}\,\langle\!\langle 2\cdots(n-1)\rangle\!\rangle\,L_{(n-1)n}}$$
$$\times \sum_{j=1}^{\lfloor(n-1)/2\rfloor-1}(-m^2_s)^j \sum_{\{w_i\}^{j-1}_{i=1}=4}^{n-3}\left(\prod_{r=1}^{j-1}\frac{\langle w_r{}^-|\slashed{K}_{(w_r+1)\cdots(w_{r+1}-1)}|w_{r+1}{}^-\rangle}{L_{w_r\cdots n}L_{(w_r+1)\cdots n}}\right)$$
$$\times \left.\langle w_1{}^+|\slashed{K}_{\hat{3}\cdots(w_1-1)}\slashed{\hat{K}}_{12}\slashed{\hat{k}}_3\slashed{k}_1|2^-\rangle\right|_{w_j=n-1}, \tag{21}$$

where we have also used the fact that $|\hat{2}^-\rangle = |2^-\rangle$ and $\langle\hat{3}^-| = \langle 3^-|$. Now,

$$\begin{aligned}\slashed{\hat{K}}_{12}\slashed{\hat{k}}_3\slashed{k}_1|2^-\rangle &= 2\hat{K}_{12}\cdot k_3\slashed{k}_1|2^-\rangle - \slashed{\hat{k}}_3\slashed{\hat{K}}_{12}\slashed{k}_1|2^-\rangle\\ &= L_{123}\slashed{k}_1|2^-\rangle - m^2_s\slashed{\hat{k}}_3|2^-\rangle, \tag{22}\end{aligned}$$

thanks to the on-shell conditions in the three-point amplitude.

Writing out the two terms, we have

$$
-\frac{i}{L_{12}\left\langle\!\left\langle 2\cdots(n-1)\right\rangle\!\right\rangle L_{(n-1)n}}\left\{\rule{0pt}{40pt}\right.
$$

$$
\sum_{j=1}^{\lfloor(n-1)/2\rfloor-1}(-m_s^2)^j\sum_{\{w_i\}_{i=1}^{j-1}=4}^{n-3}\left(\prod_{r=1}^{j-1}\frac{\langle w_r{}^-|\,\slashed{K}_{(w_r+1)\cdots(w_{r+1}-1)}\,|w_{r+1}{}^-\rangle}{L_{w_r\cdots n}L_{(w_r+1)\cdots n}}\right)\langle w_1{}^+|\,\slashed{K}_{\hat{3}\cdots(w_1-1)}\slashed{k}_1\,|2^-\rangle\Bigg|_{w_j=n-1}\quad(23)
$$

$$
+\sum_{j=2}^{\lfloor(n-1)/2\rfloor}(-m_s^2)^j\sum_{\{w_i\}_{i=2}^{j-1}=4}^{n-3}\left(\prod_{r=2}^{j-1}\frac{\langle w_r{}^-|\,\slashed{K}_{(w_r+1)\cdots(w_{r+1}-1)}\,|w_{r+1}{}^-\rangle}{L_{w_r\cdots n}L_{(w_r+1)\cdots n}}\right)\frac{\langle w_2{}^+|\,\slashed{K}_{3\cdots(w_2-1)}\slashed{k}_3\,|2^-\rangle}{L_{4\cdots n}}\Bigg|_{w_j=n-1}\left.\rule{0pt}{40pt}\right\}.
$$

We can rewrite the last factor in brackets,

$$
\frac{\langle w_2{}^+|\,\slashed{K}_{3\cdots(w_2-1)}\slashed{k}_3\,|2^-\rangle}{L_{4\cdots n}}\;=\;\frac{\langle w_1{}^-|\,\slashed{K}_{4\cdots(w_2-1)}\,|w_2{}^-\rangle}{L_{w_1\cdots n}L_{(w_1+1)\cdots n}}\langle w_1{}^+|\,\slashed{K}_{2\cdots(w_1-1)}\slashed{k}_1\,|2^-\rangle\Bigg|_{w_1=3}.\quad(24)
$$

We also note that because of the on-shell conditions on the three-point amplitude, $2\hat{k}_2\cdot k_1=0=\langle\hat{2}^-|\,\slashed{k}_1\,|2^-\rangle$, so that we can replace $\langle w_1{}^+|\,\slashed{K}_{\hat{3}\cdots(w_1-1)}\slashed{k}_1\,|2^-\rangle$ in the last factor by $\langle w_1{}^+|\,\slashed{K}_{2\cdots(w_1-1)}\slashed{k}_1\,|2^-\rangle$.

Furthermore, we can extend all but the w_1 sums in eq. (23) down to $w=3$, since the summands will necessarily vanish. The second term will then supply the $w_1=3$ terms, with some left-over pieces,

$$
-\frac{i}{L_{12}\left\langle\!\left\langle 2\cdots(n-1)\right\rangle\!\right\rangle L_{(n-1)n}}\left\{\rule{0pt}{50pt}\right.
$$

$$
\sum_{j=1}^{\lfloor(n-1)/2\rfloor-1}(-m_s^2)^j\sum_{\{w_i\}_{i=1}^{j-1}=3}^{n-3}\left(\prod_{r=1}^{j-1}\frac{\langle w_r{}^-|\,\slashed{K}_{(w_r+1)\cdots(w_{r+1}-1)}\,|w_{r+1}{}^-\rangle}{L_{w_r\cdots n}L_{(w_r+1)\cdots n}}\right)\langle w_1{}^+|\,\slashed{K}_{2\cdots(w_1-1)}\slashed{k}_1\,|2^-\rangle\Bigg|_{w_j=n-1}
$$

$$
+(-m_s^2)^j\sum_{\{w_i\}_{i=2}^{j-1}=3}^{n-3}\left(\prod_{r=1}^{j-1}\frac{\langle w_r{}^-|\,\slashed{K}_{(w_r+1)\cdots(w_{r+1}-1)}\,|w_{r+1}{}^-\rangle}{L_{w_r\cdots n}L_{(w_r+1)\cdots n}}\right)\langle w_1{}^+|\,\slashed{K}_{2\cdots(w_1-1)}\slashed{k}_1\,|2^-\rangle\Bigg|_{\substack{j=\lfloor(n-1)/2\rfloor\\ w_1=3,w_j=n-1}}
$$

$$
-(-m_s^2)^j\sum_{\{w_i\}_{i=2}^{j-1}=3}^{n-3}\langle w_1{}^+|\,\slashed{K}_{2\cdots(w_1-1)}\slashed{k}_1\,|2^-\rangle\Bigg|_{\substack{j=1\\ w_1=3,w_j=n-1}}\left.\rule{0pt}{50pt}\right\}.\quad(25)
$$

The last term is absent for $n>4$ (because the constraints are incompatible). For the penultimate term, we consider the even and odd cases separately.

If n is even,

$$
\sum_{j=1}^{\lfloor(n-1)/2\rfloor-1}\;=\;\sum_{j=1}^{\lfloor n/2\rfloor-2},\quad(26)
$$

and the penultimate term completes the sum in the first term to up to $j = \lfloor n/2 \rfloor - 1$. Note that in the penultimate term, w_1 can in any event take only the value 3, because while we have $w_r \geq 2r + 1$, the presence of $n/2 - 2$ different summation indices w_r with $w_r \geq w_{r-1} + 2$ requires that w_r take *exactly* the value $2r + 1$.

If n is odd, on the other hand, then the penultimate term in eq. (25) vanishes, because we have $(n-3)/2$ different summation indices w_r, and the last one would have to obey the incompatible constraints $w_{(n-3)/2} \leq n - 3$ and $w_{(n-3)/2} \geq n - 2$.

We thus obtain eq. (17), as desired. We have verified this equation numerically, using a set of light-cone recursion relations of the conventional kind [28], through $n = 12$.

IV.   THE $A(1_s, 2^+, \ldots, (n-2)^+, (n-1)^-, n_s)$ AMPLITUDE

We turn next to an amplitude with one negative helicity, adjacent to a scalar leg. There are two such color-ordered amplitudes, related by reflection.

Here, we shall use an $(n-1, n-2)$ shift,

$$
\begin{aligned}
|(n-1)^-\rangle &\rightarrow |(n-1)^-\rangle - z|(n-2)^-\rangle \,, \\
|(n-2)^+\rangle &\rightarrow |(n-2)^+\rangle + z|(n-1)^+\rangle \,,
\end{aligned}
\tag{27}
$$

The recursion for the target amplitude now has two terms,

$$
\begin{aligned}
A_n(1_s, 2^+, \ldots, \widehat{(n-2)}^+, &\widehat{(n-1)}^-, n_s) = \\
A_{n-1}(1_s, 2^+, \ldots, \widehat{(n-2)}^+, &\hat{K}^-_{(n-1)n\,s}) \frac{i}{K^2_{(n-1)n} - m_s^2} A_3(-\hat{K}^+_{(n-1)n\,s}, \widehat{(n-1)}^-, n_s) \\
+ A_{n-1}(1_s, 2^+, \ldots, &\hat{K}^+_{(n-3)(n-2)}, \widehat{(n-1)}^-, n_s) \\
&\times \frac{i}{K^2_{(n-3)(n-2)}} A_3(-\hat{K}^-_{(n-3)(n-2)}, (n-3)^+, \widehat{(n-2)}^+).
\end{aligned}
\tag{28}
$$

We can evaluate the first term directly,

$$
\begin{aligned}
T_1(1_s, 2^+, \ldots, &\widehat{(n-2)}^+, \widehat{(n-1)}^-, n_s) \equiv \\
A_{n-1}(1_s, 2^+, \ldots, &\widehat{(n-2)}^+, \hat{K}^-_{(n-1)n\,s}) \frac{i}{K^2_{(n-1)n} - m_s^2} A_3(-\hat{K}^+_{(n-1)n\,s}, \widehat{(n-1)}^-, n_s) \\
= -\frac{i}{L_{12} \langle\langle 1 \cdots (n-3) \rangle\rangle \, \langle (n-3)\, \widehat{(n-2)} \rangle \, L_{\widehat{(n-2)}\hat{K}_{(n-1)n}} L_{\widehat{(n-1)n}}} &\frac{\langle \widehat{(n-1)}^- | \hat{\vec{K}}_{(n-1)n} | q^- \rangle}{[q\, \widehat{(n-1)}]}
\end{aligned}
\tag{29}
$$

9

$$\times \sum_{j=1}^{\lfloor (n-1)/2 \rfloor -1} (-m_s^2)^j \sum_{\{w_i\}_{i=1}^{j-1}=3}^{n-4} \left(\prod_{r=1}^{j-1} \frac{\langle w_r^- | \slashed{K}_{(w_r+1)\cdots(w_{r+1}-1)} | w_{r+1}^- \rangle}{L_{1\cdots(w_r-1)}L_{1\cdots w_r}} \right) \langle w_1^+ | \slashed{K}_{2\cdots(w_1-1)} \slashed{k}_1 | 2^- \rangle \Bigg|_{w_j=n-1}$$

$$= -\frac{i \, \langle (n-1)^- | \slashed{k}_n | (n-2)^- \rangle^2}{L_{12} \langle\langle 2\cdots(n-3) \rangle\rangle \langle (n-3)^- | \slashed{K}_{(n-2)(n-1)} \slashed{k}_n \slashed{K}_{(n-2)(n-1)} | (n-2)^- \rangle L_{(n-2)\cdots n} L_{(n-1)n}}$$

$$\times \sum_{j=1}^{\lfloor (n-1)/2 \rfloor -1} (-m_s^2)^j \sum_{\{w_i\}_{i=1}^{j-1}=3}^{n-4} \left(\prod_{r=1}^{j-1} \frac{\langle w_r^- | \slashed{K}_{(w_r+1)\cdots(w_{r+1}-1)} | w_{r+1}^- \rangle}{L_{1\cdots(w_r-1)}L_{1\cdots w_r}} \right) \langle w_1^+ | \slashed{K}_{2\cdots(w_1-1)} \slashed{k}_1 | 2^- \rangle \Bigg|_{w_j=n-1}$$

For the second term,

$$T_2(1_s, 2^+, \ldots, \widehat{(n-2)}^+, \widehat{(n-1)}^-, n_s) \equiv$$
$$A_{n-1}(1_s, 2^+, \ldots, \hat{K}_{(n-3)(n-2)}^+, \widehat{(n-1)}^-, n_s) \tag{30}$$
$$\times \frac{i}{K_{(n-3)(n-2)}^2} A_3(-\hat{K}_{(n-3)(n-2)}^-, (n-3)^+, \widehat{(n-2)}^+),$$

we can iterate the recursion relations; suppressing all arguments but the number of legs, we obtain the following structure,

$$\begin{aligned}
A_n &= T_1(n) + \frac{iA_3}{P^2} A_{n-1} \\
&= T_1(n) + \frac{iA_3}{P^2} \Big[T_1(n-1) + \frac{iA_3}{P^2} A_{n-2} \Big] \\
&= T_1(n) + \frac{iA_3}{P^2} \Big[T_1(n-1) + \frac{iA_3}{P^2} \Big[T_1(n-2) + \frac{iA_3}{P^2} A_{n-3} \Big] \Big] \\
&\vdots
\end{aligned} \tag{31}$$

In this iteration, the $\hat{K}$ momentum at any stage will be one of the shifted momenta at the following stage; to make this explicit, it will be helpful to define,

$$\begin{aligned}
\hat{K}_{[1]}^\mu &= k_{m-1}^\mu, \\
\hat{K}_{[j]}^\mu &= \hat{K}_{m-j,\hat{K}_{[j-1]}} = k_{m-j}^\mu + \hat{K}_{[j-1]}^\mu - \frac{(k_{m-j} + \hat{K}_{[j-1]})^2 \langle m^- | \gamma^\mu | \hat{K}_{[j-1]}^- \rangle}{2 \langle m^- | \slashed{k}_{m-j} | \hat{K}_{[j-1]}^- \rangle}, \\
\widehat{\hat{K}}_{[j]}^\mu &= \hat{K}_{[j]}^\mu - \frac{(k_{m-j-1} + \hat{K}_{[j]})^2 \langle m^- | \gamma^\mu | \hat{K}_{[j]}^- \rangle}{2 \langle m^- | \slashed{k}_{m-j-1} | \hat{K}_{[j]}^- \rangle},
\end{aligned} \tag{32}$$

so that $\widehat{\hat{K}}_{[j-1]} + k_{m-j} - \hat{K}_{[j-1]} = 0$. For the specific configuration considered in this section $m = n - 1$, and hence we have

$$\begin{aligned}
\hat{K}_{[1]}^\mu &= k_{n-2}^\mu, \\
\hat{K}_{[j]}^\mu &= k_{n-j-1}^\mu + \hat{K}_{[j-1]}^\mu - \frac{(k_{n-j-1} + \hat{K}_{[j-1]})^2 \langle (n-1)^- | \gamma^\mu | \hat{K}_{[j-1]}^- \rangle}{2 \langle (n-1)^- | \slashed{k}_{n-j-1} | \hat{K}_{[j-1]}^- \rangle},
\end{aligned} \tag{33}$$

10

$$\widehat{\widehat{K}}^{\mu}_{[j]} \;=\; \hat{K}^{\mu}_{[j]} - \frac{(k_{n-j-2} + \hat{K}_{[j]})^2 \, \langle (n-1)^- | \, \gamma^\mu \, | \hat{K}_{[j]}^{\;-} \rangle}{2 \, \langle (n-1)^- | \, \slashed{k}_{n-j-2} \, | \hat{K}_{[j]}^{\;-} \rangle}.$$

Putting the arguments back into eq. (31), we obtain an explicit expression for A_n,

$$A_n(1_s, 2^+, \ldots, (n-2)^+, (n-1)^-, n_s) =$$
$$\sum_{j=1}^{n-3} \prod_{r=2}^{j} \frac{i A_3(-\hat{K}_{[r]}^{\,-}, (n-1-r)^+, \widehat{\widehat{K}}_{[r-1]}^{\,+})}{K^2_{(n-2-r)\cdots(n-2)}}$$
$$\times T_1(1_s, 2^+, \ldots, (n-2-j)^+, \hat{K}_{[j]}^{\,+}, \widehat{(n-1)}^-, n_s). \tag{34}$$

To proceed, we must simplify the product of three-point amplitudes. We can first rewrite

$$\frac{i A_3(-\hat{K}_{[2]}^{\,-}, (n-3)^+, \widehat{\widehat{K}}_{[1]}^{\,+})}{K^2_{(n-3)(n-2)}} =$$
$$\frac{[(n-3)\,\widehat{\widehat{K}}_{[1]}]^4}{[(-\hat{K}_{[2]})\,(n-3)] \, [(n-3)\,\widehat{\widehat{K}}_{[1]}] \, [\widehat{\widehat{K}}_{[1]}\,(-\hat{K}_{[2]})] \, K^2_{(n-3)(n-2)}}$$
$$= -\frac{\langle (n-1)\,(-\hat{K}_{[2]}) \rangle^2}{\langle (n-1)\,(n-3) \rangle \, \langle (n-3)\,(n-2) \rangle \, \langle (n-2)\,(n-1) \rangle}, \tag{35}$$

and then by induction, we can show that

$$\prod_{r=2}^{j} \frac{i A_3(-\hat{K}_{[r]}^{\,-}, (m-r)^+, \widehat{\widehat{K}}_{[r-1]}^{\,+})}{K^2_{(m-r)\cdots(m-1)}} =$$
$$-\frac{\langle m\,(-\hat{K}_{[j]}) \rangle^2}{\langle m\,(m-j+1) \rangle \, \langle\!\langle (m-j+1)\cdots m \rangle\!\rangle}. \tag{36}$$

where for the case considered in this section $m = n - 1$.

Using this result, we obtain our final formula for the amplitude,

$$A(1_s, 2^+, \ldots, (n-2)^+, (n-1)^-, n_s) =$$
$$\frac{i \, \langle (n-1)^- | \, \slashed{k}_1 \slashed{k}_n \, | (n-1)^+ \rangle^2}{\langle\!\langle 2 \cdots (n-1) \rangle\!\rangle \, K^2_{2\ldots(n-1)} \, \langle (n-1)^- | \, \slashed{K}_{2\ldots(n-1)} \slashed{k}_1 \, | 2^+ \rangle}$$
$$+ \sum_{l=1}^{n-4} \frac{i \, \langle (n-l-2)\,(n-l-1) \rangle}{\langle\!\langle 2 \cdots (n-1) \rangle\!\rangle \, L_{12} \, K^2_{(n-l-1)\cdots(n-1)} \, L_{(n-l-1)\cdots n}} \tag{37}$$
$$\times \frac{\langle (n-1)^- | \, \slashed{K}_{(n-l-1)\cdots(n-1)} \slashed{k}_n \, | (n-1)^+ \rangle^2}{\langle (n-l-2)^- | \, \slashed{K}_{(n-l-1)\cdots(n-1)} \slashed{k}_n \, | (n-1)^+ \rangle \, \langle (n-l-1)^- | \, \slashed{K}_{(n-l-1)\cdots(n-1)} \slashed{k}_n \, | (n-1)^+ \rangle}$$
$$\times \sum_{j=1}^{\lfloor (n-l)/2 \rfloor - 1} (-m_s^2)^j \sum_{\substack{\{w_i\}_{i=1}^{j-1} \\ w_i = w_{i-1}+2}}^{n-l-3} \left(\prod_{r=1}^{j-1} \frac{\langle w_r^- | \, \slashed{K}_{(w_r+1)\cdots(w_{r+1}-1)} \, | w_{r+1}^- \rangle}{L_{1\cdots(w_r-1)} L_{1\cdots w_r}} \right)$$
$$\times \langle w_1^+ | \, \slashed{K}_{2\cdots(w_1-1)} \slashed{k}_1 \, | 2^- \rangle \Bigg|_{\substack{\langle w_j^+ | = \langle (n-1)^- | \slashed{K}_{(n-l-1)\cdots(n-1)}, \\ w_0 = 1, \, w_j = n-l-1}}$$

11

where we have separated out the m-independent terms. The reader can readily verify that these reproduce the required maximally helicity-violating (MHV) amplitude in the massless limit.

We can rewrite eq. (37) in a slightly more compact form as

$$
\begin{aligned}
A(1_s, 2^+, \ldots, (n-2)^+, (n-1)^-, n_s) = \\
\frac{i \langle (n-1)^- | \not{k}_1 \not{k}_n | (n-1)^+ \rangle^2}{\langle\langle 2 \cdots (n-1) \rangle\rangle K_{2\cdots(n-1)}^2 \langle (n-1)^- | \not{K}_{2\cdots(n-1)} \not{k}_1 | 2^+ \rangle} \\
+ \sum_{l=1}^{n-4} \frac{i \langle (n-l-2)(n-l-1) \rangle}{\langle\langle 2 \cdots (n-1) \rangle\rangle L_{12} K_{(n-l-1)\cdots(n-1)}^2 L_{(n-l-1)\cdots n}} \\
\times \frac{\langle (n-1)^- | \not{K}_{(n-l-1)\cdots(n-1)} \not{k}_n | (n-1)^+ \rangle^2 G(2, n-l-1; n-1)}{\langle (n-l-2)^- | \not{K}_{(n-l-1)\cdots(n-1)} \not{k}_n | (n-1)^+ \rangle \langle (n-l-1)^- | \not{K}_{(n-l-1)\cdots(n-1)} \not{k}_n | (n-1)^+ \rangle}
\end{aligned}
\tag{38}
$$

by introducing

$$
\begin{aligned}
G(a_1, b_1; m) = \sum_{j=1}^{\lfloor (b_1-a_1+1)/2 \rfloor} (-m_s^2)^j \sum_{\substack{\{w_i\}_{i=1}^{j-1} \\ w_i = w_{i-1}+2}} \left(\prod_{r=1}^{j-1} \frac{\langle w_r^- | \not{K}_{(w_r+1)\cdots(w_{r+1}-1)} | w_{r+1}^- \rangle}{L_{1\cdots(w_r-1)} L_{1\cdots w_r}} \right) \\
\times \langle w_1^+ | \not{K}_{a_1 \cdots (w_1-1)} \not{k}_1 | a_1^- \rangle \Bigg|_{\substack{\langle w_j^+ | = \langle m^- | \not{K}_{b_1 \cdots m}, \\ w_0 = a_1-1, w_j = b_1}}
\end{aligned}
\tag{39}
$$

In the four-point case, only the first term in eq. (37) is present, and we obtain,

$$
\begin{aligned}
A_4(1_s, 2^+, 3^-, 4_s) &= i \frac{\langle 3^- | \not{k}_1 \not{k}_4 | 3^+ \rangle^2}{\langle 2\,3 \rangle K_{23}^2 \langle 3^- | \not{k}_2 \not{k}_1 | 2^+ \rangle} \\
&= -i \frac{\langle 3^- | \not{k}_1 | 2^- \rangle^2}{K_{23}^2 L_{12}};
\end{aligned}
\tag{40}
$$

in the five-point case,

$$
- \frac{i \langle 4^- | \not{k}_1 \not{K}_{23} | 4^+ \rangle^2}{K_{234}^2 \langle 2\,3 \rangle \langle 3\,4 \rangle \langle 2^- | \not{k}_1 \not{K}_{23} | 4^+ \rangle} - \frac{im_s^2 \langle 4^- | \not{k}_5 | 3^- \rangle^2 [2\,3]}{L_{12} L_{45} [3\,4] \langle 4^- | \not{k}_5 \not{K}_{34} | 2^+ \rangle};
\tag{41}
$$

and in the six-point case,

$$
\begin{aligned}
- \frac{i \langle 5^- | \not{k}_1 \not{K}_{234} | 5^+ \rangle^2}{K_{2345}^2 \langle 2\,3 \rangle \langle 3\,4 \rangle \langle 4\,5 \rangle \langle 2^- | \not{k}_1 \not{K}_{234} | 5^+ \rangle} + \frac{im_s^2 \langle 5^- | \not{k}_6 | 4^- \rangle^2 \langle 4^+ | \not{K}_{23} \not{k}_1 | 2^- \rangle}{L_{12} L_{456} L_{56} \langle 2\,3 \rangle [4\,5] \langle 5^- | \not{k}_6 \not{K}_{45} | 3^+ \rangle} \\
- \frac{im_s^2 \langle 2\,3 \rangle \langle 5^- | \not{k}_6 \not{K}_{34} | 5^+ \rangle^2 \langle 5^- | \not{K}_{34} | 2^- \rangle}{K_{345}^2 L_{12} \langle 2\,3 \rangle \langle 3\,4 \rangle \langle 4\,5 \rangle \langle 5^- | \not{k}_6 \not{K}_{345} | 2^+ \rangle \langle 5^- | \not{k}_6 \not{K}_{45} | 3^+ \rangle}
\end{aligned}
\tag{42}
$$

in agreement with ref. [9] up to an irrelevant overall phase[1].

---

[1] After correcting the sign in the second term of eqn. (3.18) of ref. [9].

We have verified that eq. (37) has the correct collinear limits, and have also verified it numerically against a set of light-cone recurrence relations up to $n = 12$.

V.   THE $A(1_s, 2^+, \ldots, m^-, (m+1)^+, \ldots, n_s)$ AMPLITUDE

In this section, we will generalize the result from the previous section, and obtain an expression for the amplitude with a negative-helicity gluon not color-adjacent to one of the massive scalar legs. Here, we will use an $(m, m-1)$ shift,

$$
\begin{aligned}
|m^-\rangle &\rightarrow |m^-\rangle - z\,|(m-1)^-\rangle, \\
|(m-1)^+\rangle &\rightarrow |(m-1)^+\rangle + z\,|m^+\rangle,
\end{aligned}
\tag{43}
$$

Our target amplitude is then given by the following recursive form

$$
\begin{aligned}
&A_n(1_s, 2^+, \ldots, m^-, (m+1)^+, \ldots, (n-1)^+, n_s) \\
&= A_m(1_s, 2^+, \ldots, \widehat{(m-1)}^+, -\hat{K}^-_{1\cdots(m-1),s}) \frac{1}{K^2_{1\cdots(m-1)} - m_s^2} \\
&\qquad \times A_{n-m+2}(\hat{K}^+_{1\cdots(m-1),s}, \hat{m}^-, (m+1)^+, \ldots, n_s) \\
&\quad + \sum_{i=m+1}^{n-1} A_{n+m-i}(1_s, 2^+, \ldots, \widehat{(m-1)}^+, \hat{K}^+_{(i+1)\cdots(m-1)}, (i+1)^+, \ldots, n_s) \frac{1}{K^2_{(i+1)\cdots(m-1)}} \\
&\qquad\qquad \times A_{i-m+2}(\hat{K}^-_{(i+1)\cdots(m-1)}, \hat{m}^-, (m+1)^+, \ldots, i^+) \\
&\quad + A_3((m-2)^+, \widehat{(m-1)}^+, -\hat{K}^-_{(m-2)(m-1)}) \frac{1}{K^2_{(m-2)(m-1)}} \\
&\qquad\qquad \times A_{n-1}(1_s, 2^+, \ldots, (m-3)^+, \hat{K}^+_{(m-2)(m-1)}, \hat{m}^-, \ldots, n_s) \\
&\equiv U_1(1_s, 2^+, \ldots, m^-, \ldots, (n-1)^+, n_s) + A_3((m-2)^+, \widehat{(m-1)}^+, -\hat{K}^-_{(m-2)(m-1)}) \\
&\quad \times \frac{1}{K^2_{(m-2)(m-1)}} A_{n-1}(1_s, 2^+, \ldots, (m-3)^+, \hat{K}^+_{(m-2)(m-1)}, \hat{m}^-, \ldots, n_s^+)
\end{aligned}
\tag{44}
$$

where as indicated U_1 is defined as the first two terms. It depends only on gluonic amplitudes, all-plus massive-scalar amplitudes and $m = 1$ lone-negative-helicity massive-scalar amplitudes. We obtained all-multiplicity solutions for these in previous sections, and from them we can obtain the all-n form of this function. The U_1 term corresponds to the $l_1 = 2$ terms in eq. (47).

For the last term in eq. (44) we can, as before, iterate the recurrence relation eq. (44), whereupon we obtain

$$
A_n(1_s, 2^+, \ldots, m^-, (m+1)^+, \ldots, (n-1)^+, n_s)
$$

13

$$= \sum_{j=1}^{m-2} \prod_{r=2}^{j} \frac{iA_3(-\hat{K}_{[r]}^-, (m-r)^+, \hat{\bar{K}}_{[r-1]}^+)}{K_{(m-r)\cdots(m-1)}^2}$$
$$\times U_1(1_s^+, 2^+, \ldots, (m-j-1)^+, \hat{K}_{[j]}^+, \hat{m}^-, \ldots, n_s), \tag{45}$$

where $\hat{K}_{[r]}$ and $\hat{\bar{K}}_{[r]}$ are as defined in eq. (32), and where we have used

$$A_{n-m+3}(1_s, \hat{K}_{[m-2]}^+, \hat{m}^-, \ldots, n_s) = U_1(1_s, \hat{K}_{[m-2]}^+, \hat{m}^-, \ldots, n_s). \tag{46}$$

The amplitude we seek is now defined entirely in terms of U_1. Using this iterated recurrence form of the amplitude along with the explicit form of U_1 and also eq. (36) we arrive at our final form for the amplitude,

$$A_n(1_s, 2^+, \ldots, m^-, \ldots, (n-1)^+, n_s) = -\frac{i}{\langle\!\langle 2 \cdots (n-1)\rangle\!\rangle}$$
$$\times \sum_{l_1=2}^{m-1} \sum_{l_2=m}^{n-2} \left(\frac{f_0(n, m; l_1)\overline{f}_0(n, m; l_2)}{\langle m^-|\,\slashed{K}_{1\cdots m}\slashed{K}_{2\cdots(l_2+1)} + \slashed{K}_{2\cdots m}\slashed{k}_1\,|(l_2+1)^+\rangle} \right.$$
$$\times \frac{\langle m^-|\,\slashed{K}_{1\cdots m}\slashed{K}_{(m-l_1+1)\cdots m}\,|m^+\rangle^2 \langle m^-|\,\slashed{K}_{1\cdots m}\slashed{K}_{m\cdots(l_2+1)}\,|m^+\rangle^2}{\langle m^-|\,\slashed{K}_{m\cdots(n-1)}\slashed{K}_{(m-l_1+1)\cdots n} + \slashed{k}_n\slashed{K}_{(m-l_1+1)\cdots(n-1)}\,|(m-l_1+1)^+\rangle} \tag{47}$$
$$\times \frac{1}{\langle m^-|\,\slashed{K}_{m\cdots(l_2+1)}(\slashed{K}_{(l_2+2)\cdots(n-1)}\slashed{K}_{(m-l_1+1)\cdots n} + \slashed{k}_n\slashed{K}_{(m-l_1+1)\cdots(n-1)})\slashed{K}_{(m-l_1+1)\cdots m}\,|m^+\rangle}$$
$$+ \frac{f_1(n, m, l_1, l_2+1; l_1)f_2(n, m, l_1, l_2; l_2+1)\langle m^-|\,\slashed{K}_{m\cdots(l_2+1)}\slashed{K}_{(m-l_1+1)\cdots m}\,|m^+\rangle^4}{K_{(m-l_1+1)\cdots(l_2+1)}^2 \langle(l_2+1)^-|\,\slashed{K}_{(m-l_1+1)\cdots(l_2+1)}\slashed{K}_{(m-l_1+1)\cdots m}\,|m^+\rangle}$$
$$\left. \times \frac{1}{\langle m^-|\,\slashed{K}_{m\cdots(l_2+1)}\slashed{K}_{(m-l_1+1)\cdots(l_2+1)}\,|(m-l_1+1)^+\rangle} \right),$$

which is valid for $2 < m < n-1$. (For $m = n-1$, one should use either eq. (37) or (38); for $m = 2$, one should use those formulæ after reflection.) In the above equation,

$$f_0(n, m; i) = \begin{cases} -1, & i = m-1; \\ \dfrac{\langle(m-i)(m-i+1)\rangle G(2, m-i+1; m)/L_{(m-i+1)\cdots n}}{L_{12}\langle(m-i)^-|\,\slashed{K}_{(m-i+1)\cdots(n-1)}\slashed{K}_{m\cdots n} + \slashed{k}_n\slashed{K}_{m\cdots(n-1)}\,|m^+\rangle}, & 2 \le i < m-1. \end{cases} \tag{48}$$

$$\overline{f}_0(n, m; i) = \begin{cases} 1, & i = n-2; \\ \dfrac{\langle(i+1)(i+2)\rangle \overline{G}(n; i+1, n-1; m)}{L_{1\cdots(i+1)}\langle(i+2)^-|\,\slashed{K}_{2\cdots(i+1)}\slashed{K}_{1\cdots m} + \slashed{k}_1\slashed{K}_{2\cdots m}\,|m^+\rangle L_{n-1,n}}, & m \le i < n-2. \end{cases}$$

and

$$f_1(n, m, l_1, l_2; i) = \begin{cases} \dfrac{G_0(2, m-l_1+1; l_2, n-1; 2; m)}{\langle m^-|\slashed{K}_{m\cdots l_2}\slashed{K}_{2\cdots l_2}\slashed{k}_1\slashed{K}_{2\cdots m}|m^+\rangle}, & i = m-1; \\ \dfrac{\langle(m-i)(m-i+1)\rangle G_1(2, m-i+1; l_2, n-1; 2; m)}{L_{12}\langle(m-i)^-|\slashed{K}_{(m-i+1)\cdots l_2}\slashed{K}_{m\cdots l_2}|m^+\rangle}, & 2 \le i < m-1. \end{cases}$$

14

$$
\begin{equation}
\tag{49}
\end{equation}
$$

$$
f_2(n,m,l_1,l_2;i) = \begin{cases} \dfrac{-1}{\langle m^-|\mathbb{K}_{m\cdots(n-1)}\not{k}_n\mathbb{K}_{(m-l_1+1)\cdots(n-1)}\mathbb{K}_{(m-l_1+1)\cdots m}|m^+\rangle}, & i = n-1; \\[2ex] \dfrac{-\langle i\,(i+1)\rangle}{L_{(n-1)n}\langle (i+1)^-|\mathbb{K}_{(m-l_1+1)\cdots i}\mathbb{K}_{(m-l_1+1)\cdots m}|m^+\rangle}, & m+1 \le i < n-1. \end{cases}
$$

The attentive reader will notice that the dimensionality of the expressions for f_1 and f_2 is different in each of the two cases. Nonetheless, all terms in eq. (47) have the same dimension, because this difference is compensated by the inapplicability of the first replacement in eq. (54) when $l_2 = n - 2$.

We have also defined

$$
\begin{aligned}
G_u(a_1,b_1;a_2,b_2;c;m) = & \\
\sum_{j=1}^{\lfloor(b_1+b_2-(a_1+a_2)+c)/2\rfloor} (-m_s^2)^j & \sideset{}{'}\sum_{\substack{\{w_i\}_{i=1}^{j-1}\\w_i\in S_i}} \left(\prod_{r=1}^{j-1} \frac{\langle w_r^-|\mathbb{K}_{(w_r+1)\cdots(w_{r+1}-1)}|w_{r+1}^-\rangle}{L_{1\cdots(w_r-1)}L_{1\cdots w_r}}\right)
\end{aligned}
\tag{50}
$$

$$
\times \left.\begin{cases} \langle w_1^+|\,\mathbb{K}_{a_1\cdots(w_1-1)}\not{k}_1\mathbb{K}_{2\cdots m}\,|m^+\rangle, & u=0, \\[1ex] \langle w_1^+|\,\mathbb{K}_{a_1\cdots(w_1-1)}\not{k}_1\,|a_1^-\rangle, & u=1, \end{cases}\right\}\Bigg|_{\substack{w_0=a_1-1\\w_j=b_2}},
$$

where the sum for each w_i is over a set of momenta S_i, defined as follows. Define the set $S_0 = \{a_1,\ldots,b_1,a_2,\ldots,b_2\}$, let $\operatorname{succ}(a)$ be the element following a, and $\operatorname{pred}(a)$ the element preceeding a in S_0. The set S_1 is then defined by omitting the first and last two elements of S_0,

$$
S_1 = \{\operatorname{succ}(a_1),\ldots,b_1,a_2,\ldots,\operatorname{pred}(\operatorname{pred}(b_2))\}.
\tag{51}
$$

Each subsequent set S_i depends on the value of w_{i-1}; it will contain all elements in $S_0 = \{a_1,\ldots,b_1,a_2,\ldots,b_2\}$ following the element after w_{i-1}, and through two elements prior to b_2. This is the generalization of the sum for w_i starting at $w_{i-1}+2$ (and ending at $w_{i+1}-2$). That is,

$$
S_i = \{\operatorname{succ}(\operatorname{succ}(w_{i-1})),\ldots,\operatorname{pred}(\operatorname{pred}(b_2))\}.
\tag{52}
$$

Note, however, that the sums of momenta $K_{i\cdots j}$ are over consecutive momenta, *not* restricted to S_0.

The prime signifies that in addition, we must make the following replacements whenever w_k is inside a bra or a ket. For $w_k = b_1$,

$$
\frac{\langle w_k^-|\mathbb{K}_{(w_k+1)\cdots(w_{k+1}-1)}}{L_{1\cdots w_k}} \rightarrow \frac{\langle m^-|\mathbb{K}_{m\cdots a_2}\mathbb{K}_{b_1\cdots a_2}\mathbb{K}_{w_k\cdots(w_{k+1}-1)}}{\langle m^-|\mathbb{K}_{b_1\cdots m}\left(\mathbb{K}_{1\cdots(b_1-1)}\mathbb{K}_{2\cdots a_2} + \mathbb{K}_{2\cdots(b_1-1)}\not{k}_1\right)\mathbb{K}_{m\cdots a_2}|m^+\rangle}
$$

$$
\mathbb{K}_{(w_{k-1}+1)\cdots(w_k-1)}|w_k^-\rangle \rightarrow -\mathbb{K}_{(w_{k-1}+1)\cdots(w_k-1)}\mathbb{K}_{b_1\cdots m}|m^+\rangle
\tag{53}
$$

15

and for $w_k = a_2$,

$$\frac{\langle w_k^-|\slashed{K}_{(w_k+1)\cdots(w_{k+1}-1)}}{L_{1\cdots(w_k-1)}} \;\to\; \frac{\langle m^-|\slashed{K}_{b_1\cdots m}\slashed{K}_{b_1\cdots a_2}\slashed{K}_{(w_k+1)\cdots(w_{k+1}-1)}}{\langle m^-|\slashed{K}_{b_1\cdots m}\left(\slashed{K}_{1\cdots(b_1-1)}\slashed{K}_{2\cdots a_2} + \slashed{K}_{2\cdots(b_1-1)}\slashed{k}_1\right)\slashed{K}_{m\cdots a_2}|m^+\rangle}$$

$$\slashed{K}_{(w_{k-1}+1)\cdots(w_k-1)}|w_k^-\rangle \;\to\; \slashed{K}_{(w_{k-1}+1)\cdots w_k}\slashed{K}_{m\cdots a_2}|m^+\rangle \tag{54}$$

These replacements should also be applied to $\langle w_1^+|$ in the last factor. That is if $w_1 = b_1$,

$$\langle w_1^+|\slashed{K}_{a_1\cdots(w_1-1)} \;\to\; \langle m^-|\slashed{K}_{b_1\cdots m}\slashed{K}_{a_1\cdots(w_1-1)}, \tag{55}$$

and if $w_1 = a_2$,

$$\langle w_1^+|\slashed{K}_{a_1\cdots(w_1-1)} \;\to\; -\langle m^-|\slashed{K}_{m\cdots a_2}\slashed{K}_{a_1\cdots w_1}. \tag{56}$$

This definition of G is a generalization of that given in eq. (39), the two are related via $G(a_1, b_1; m) \equiv G_1(a_1, b_1; b_1, b_1; 1; m)$ as $\{a_1, \dots, b_1\} \bigcup \{b_1\} = \{a_1, \dots, b_1\}$ and only the second replacement from eq. (53) is relevant. Similarly $\overline{G}$ is defined as

$$\overline{G}(n; a_2, b_2; m) \;=\; \sum_{j=1}^{\lfloor (b_2-a_2+1)/2 \rfloor} (-m_s^2)^j \sum_{\substack{\{w_i\}_{i=1}^{j-1} = a_2+2 \\ w_i = w_{i+1}+2}}^{b_2-1} \left(\prod_{r=1}^{j-1} \frac{\langle w_r^-|\slashed{K}_{(w_{r+1}+1)\cdots(w_r-1)}|w_{r+1}^-\rangle}{L_{w_r\cdots n}L_{(w_r+1)\cdots n}} \right) \tag{57}$$

$$\times \left. \left\langle w_1^+\right| \slashed{K}_{(w_1+1)\cdots(n-1)}\slashed{k}_n \left|b_2^-\right\rangle \right|_{\substack{\langle w_j^+| = \langle m^-|\slashed{K}_{m\cdots a_2} \\ w_j = a_2}}.$$

There is only a single massless contribution in eq. (47); it arises from the $(l_1 = m-1, l_2 = n - 2)$ term and is given by

$$\frac{i \langle m^-|\slashed{k}_1\slashed{K}_{2\cdots m}|m^+\rangle^2}{\langle\langle 2\cdots(n-1)\rangle\rangle \langle m^-|\slashed{K}_{m\cdots(n-1)}\slashed{k}_n|(n-1)^+\rangle}$$

$$\times \frac{\langle m^-|\slashed{k}_n\slashed{K}_{m\cdots(n-1)}|m^+\rangle^2}{\langle m^-|\slashed{K}_{m\cdots(n-1)}\slashed{k}_n\slashed{K}_{2\cdots(n-1)}\slashed{K}_{2\cdots m}|m^+\rangle \langle m^-|\slashed{K}_{2\cdots m}\slashed{k}_1|2^+\rangle} \tag{58}$$

which is readily seen to reduce to the expected MHV amplitude in the massless limit. Using eq. (47) we can derive the form of the amplitude in the five point, $m = 3$ case

$$A_5(1_s, 2^+, 3^-, 4^+, 5_s)$$
$$= -\frac{i \langle 3^-|\slashed{k}_1|2^-\rangle^2 \langle 3^-|\slashed{k}_5|4^-\rangle^2}{L_{12} \langle 2\,3\rangle \langle 3\,4\rangle L_{45} \langle 4^+|\slashed{k}_5\slashed{K}_{34}|2^-\rangle} + \frac{im_s^2 [2\,4]^4}{K_{2\cdots4}^2 [2\,3][3\,4] \langle 4^+|\slashed{k}_5\slashed{K}_{34}|2^-\rangle}, \tag{59}$$

this result matches exactly that given in ref. [9]. For the six point case with $m = 4$ we have

$$A_s(1_s, 2^+, 3^+, 4^-, 5^+, 6_s) =$$

16

$$\frac{i\langle 4^-|\,\slashed{k}_1\slashed{K}_{23}\,|4^+\rangle^2\,\langle 4^-|\,\slashed{k}_6\,|5^-\rangle^2}{\langle 2\,3\rangle\langle 3\,4\rangle\langle 4\,5\rangle\,L_{56}\,\langle 5^+|\,\slashed{k}_6\slashed{K}_{2\cdots5}\slashed{K}_{23}\,|4^+\rangle\,\langle 4^-|\,\slashed{K}_{23}\slashed{k}_1\,|2^+\rangle}$$
$$-\frac{im_s^2\,[2\,3]\,\langle 4^-|\,\slashed{K}_{56}\,|3^-\rangle^2\,\langle 4^-|\,\slashed{k}_6\,|5^-\rangle^2}{L_{12}L_{4\cdots6}L_{56}\,\langle 4\,5\rangle\,\langle 2^-|\,\slashed{k}_1\slashed{K}_{23}\,|4^+\rangle\,\langle 5^+|\,\slashed{k}_6\slashed{K}_{45}\,|3^-\rangle}$$
$$-\frac{im_s^2\,[3\,5]^4\,\langle 5^+|\,\slashed{K}_{2\cdots4}\slashed{k}_1\,|2^-\rangle}{L_{12}\,[3\,4]\,[4\,5]\,\langle 2^-|\,\slashed{K}_{34}\,|5^-\rangle\,\langle 5^+|\,\slashed{k}_6\slashed{K}_{45}\,|3^-\rangle\,K_{3\cdots5}^2}$$
$$-\frac{im_s^2\,\langle 4^-|\,\slashed{K}_{23}\,|5^-\rangle^4}{K_{2\cdots5}^2 K_{2\cdots4}^2\,\langle 2\,3\rangle\langle 3\,4\rangle\,\langle 5^+|\,\slashed{k}_6\slashed{K}_{2\cdots5}\slashed{K}_{23}\,|4^+\rangle\,\langle 2^-|\,\slashed{K}_{34}\,|5^-\rangle}. \tag{60}$$

Although this result has a different form from that given in ref. [9], we have checked that the two agree numerically up to an overall phase[2]. We have also compared the result eq. (47) numerically to amplitudes computed via light-cone recurrence relations, and found complete agreement through $n = 12$.

VI.   CONCLUSIONS

On-shell recursion relations have emerged as a powerful method for deriving analytic expressions for tree-level amplitudes. In this paper, we have given all-multiplicity solutions to the BGKS recursion relation for amplitudes with a massive scalar pair for the two simplest helicity configurations. These are the amplitudes with an arbitrary number of positive-helicity gluons, and either no or one gluon of negative helicity. (Of course, the corresponding amplitudes with an arbitrary number of negative-helicity gluons, and up to one gluon of positive helicity, can be obtained by spinor conjugation.) We have checked the principal results, eqs. (12), (37), and (47), against a numerical implementation of the older light-cone recursion relations, through $n = 12$.

The primary application of these results is to computations of one-loop QCD amplitudes within the unitarity-based method. The massive scalar amplitudes are equivalent to amplitudes with the scalar legs computed fully in D dimensions. These in turn can be used to obtain rational terms in amplitudes, which have no cuts in four dimensions, but do have cuts in D dimensions. They should also be useful for proving general factorization properties in complex momenta upon which the recursion-relations part of the unitarity-bootstrap combined approach relies [30]. For such applications, a relatively compact analytic form is desirable, and the recursion relations allow one to obtain such a form.

---

[2] After correcting a typographical error in eqn. (3.21) of ref. [9].

While the all-plus amplitude did require guesswork for an all-n ansatz, it is interesting to note that the amplitudes with a negative-helicity gluon could then be derived without having to guess an ansatz for the general form. This would not be the case using the older, off-shell recursion relations [27, 28]. While the over-all complexity of the amplitudes would increase with increasing number of negative-helicity gluons, the same approach of unwinding three-point vertices could be used in such cases as well. These too would not require new ansätze for massive-scalar amplitudes. The calculations in this paper also illustrate the extent to which calculations beyond those in purely massless gauge theories are feasible. We would expect, for example, that amplitudes with massive quarks or vector bosons instead of massive scalars would yield to a calculation of similar nature.

Acknowledgments

APPENDIX A: COMPUTATIONAL COMPLEXITY OF BERENDS–GIELE RE-CURSION RELATIONS

In order to use QCD amplitudes in evaluating experimental data, one must evaluate them numerically. How many operations are required to do so? Even at tree level, each amplitude, and indeed each different helicity amplitude, require a factorial number of Feynman diagrams. This might seem to imply that a factorial number of operations are required; as we shall show here, a better organization of the calculation can dramatically reduce this complexity.

The complete squared amplitude can be written in terms of color-ordered amplitudes. At leading order in the number of colors, which is all we will consider here, computing the amplitude does require a sum over a factorial number of color orderings. However, for integrated quantities, one can use the symmetry of final-particle phase space to reduce this to a sum over a linear number of orderings. We are thus left to consider the cost of computing a given color-ordered amplitude.

There are $\mathcal{O}(2^{n/2})$ independent helicity amplitudes, so clearly the complexity of computing

all of them must scale at least exponentially in the number of external legs. There are an exponential number of color-ordered Feynman diagrams for any given helicity amplitude, which would suggest that we still require an exponential number of operations.

In this appendix, we show that the number of operations required to evaluate a typical helicity amplitude is only polynomial in the number of external legs, so long as the calculation is organized properly. To do so, we will make use of a light-cone version [28, 33] of the Berends–Giele recursion relations [27]. (The complexity is polynomial for any version of this type of off-shell recursion relation, suitably organized; only the prefactors differ.) In our estimates, we take each basic operation — addition, multiplication, division — to have constant complexity, that is $\mathcal{O}(1)$.

Let $J^{\pm}(1, \ldots, n)$ denote the (amputated) current with n on-shell external legs, and one off-shell external leg of the given helicity (defined using a light-cone gauge vector q). We can write a recursion relation for this current,

$$
\begin{aligned}
J^{\sigma} \ (1, \ldots, n) = \\
-\sum_{j_1=1}^{n-1} \sum_{\sigma_{1,2}=\pm} \frac{V_3^{\sigma\sigma_1\sigma_2}}{K_{1\ldots j_1}^2 K_{(j_1+1)\ldots n}^2} J^{-\sigma_1}(1, \ldots, j_1) J^{-\sigma_2}(j_1+1, \ldots, n) \\
+\sum_{j_1=1}^{n-2} \sum_{j_2=j_1+1}^{n-1} \sum_{\sigma_{1,2,3}=\pm} \frac{iV_4^{\sigma\sigma_1\sigma_2\sigma_3}}{K_{1\ldots j_1}^2 K_{(j_1+1)\ldots j_2}^2 K_{(j_2+1)\ldots n}^2} J^{-\sigma_1}(1, \ldots, j_1) \\
\times J^{-\sigma_2}(j_1+1, \ldots, j_2) J^{-\sigma_3}(j_2+1, \ldots, n)
\end{aligned}
\tag{A1}
$$

In order to compute an n-point amplitude, A_n, we compute $J^{\sigma}(1, \ldots, n-1)$ with the n-th leg on-shell instead of off-shell. This in turn requires the computation of currents with fewer on-shell external legs. We can organize this computation as follows: first compute all the required three-point currents; then the four-point currents, and so on through the desired n-point current. At each stage, only previously-computed lower-point currents appear, so we need compute only the vertices and propagators, then multiply factors and perform the sums. In practice, it is probably more convenient to use caching rather than precomputation. That is, use the recursive formula, but during the recursive descent, record the (numerical) value of each newly-computed current. The next time that current is required, use the previously-computed value instead of computing it anew.

For each j-point current contributing to the final n-point current, the computation is clearly dominated by the four-point vertices and associated double sums. We can also

precompute all momentum sums that appear in our computation. These are always sums of consecutive momenta, so there are $\mathcal{O}(n^2)$ different ones, and this precomputation is of $\mathcal{O}(n^3)$ in operations. We can also precompute required Lorentz and spinor products, which requires $\mathcal{O}(n^2)$ operations. Each four-point vertex then takes a constant number of operations to compute. There are $\mathcal{O}(j^2)$ different terms, each requiring a constant number of operations, and so the computation of each j-point current requires $\mathcal{O}(j^2)$ additional operations.

For a given j, we need $n - j + 1$ different j-point currents: $J(1, \ldots, j - 1)$ through $J(n - j + 1, n - 1)$. The overall computational complexity is thus of order,

$$\sum_{j=3}^{n} (n - j + 1) j^2 \sim \mathcal{O}(n^4), \tag{A2}$$

which is indeed polynomial.

This is possible because we can re-use information: the computation of J_{17}, for example, requires knowledge of both $J_{10}(1, \ldots, 10)$ and $J_5(1, \ldots, 5)$. The computation of $J_{10}(1, \ldots, 10)$ in turn requires knowledge of $J_5(1, \ldots, 5)$, but this five-point current is the *same* as the one appearing in the top-level sum for J_{17}, and thus need be computed only once. This saving would not be possible if we insisted on writing out an analytic expression for J_{17} in terms of spinor products, because the expression for J_5 would then appear (and effectively be computed) many times.

We can obtain a more precise count of the number of operations as follows. Denote the number of operations required to compute a three-vertex V_3 by c_3, and that for a four-vertex V_4 by c_4. If all helicity arguments to a three-vertex are identical, it vanishes, so for a given choice of the current's helicity σ, there are only three different helicity choices we must compute. A straightforward organization of the sums requires four multiplications or divisions and one addition for each one, for example,

$$\frac{V^{-+-} J_1^- J_2^+}{K_1^2 K_2^2} + \frac{V^{--+} J_1^+ J_2^-}{K_1^2 K_2^2} + \frac{V^{-++} J_1^- J_2^-}{K_1^2 K_2^2}, \tag{A3}$$

where $J_1 \equiv J(1, \ldots, j_1)$, $K_1 \equiv K_{1 \ldots j_1}$, etc. It is possible to reduce this slightly by combining terms as follows,

$$\frac{1}{K_1^2 K_2^2} [V^{-+-} J_1^- J_2^+ + J_2^- (V^{--+} J_1^+ + V^{-++} J_1^-)], \tag{A4}$$

giving instead a count of $3c_3 + 10$ for a typical term, or $(3c_3 + 10)(j - 2)$ for a j-point current. (This is still an overestimate of the optimal count, because we ignore the reduction due to terms containing a two-point current as it depends on the choice of external helicities.)

20

Only four-vertices with two negative-helicity and two positive-helicity arguments are non-vanishing, so here we again have three different helicity configurations for each choice of the current's helicity. This would give six multiplications and one addition for each term in the double sum over j_1 and j_2 and $3(c_4 + 7)(j - 2)(j - 3)/2$ overall. Here, we can reorganize not only each term but the double sum as well,

$$
\sum_{j_1=1}^{j-3} \sum_{j_2=j_1+1}^{j-2} \left[\frac{V^{-++-} J_1^- J_2^- J_3^+}{K_1^2 K_2^2 K_3^2} + \frac{V^{-+-+} J_1^- J_2^+ J_3^-}{K_1^2 K_2^2 K_3^2} + \frac{V^{--++} J_1^+ J_2^- J_3^-}{K_1^2 K_2^2 K_3^2} \right] =
$$
$$
\sum_{j_1=1}^{j-3} \frac{1}{K_1^2} \left\{ J_1^- \sum_{j_2=j_1+1}^{j-2} \frac{1}{K_2^2 K_3^2} [V^{-++-} J_2^- J_3^+ + V^{-+-+} J_2^+ J_3^-] + J_1^+ \sum_{j_2=j_1+1}^{j-2} \frac{V^{--++} J_2^- J_3^-}{K_2^2 K_3^2} \right\},
$$
$$
\text{(A5)}
$$

giving an overall operation count of

$$
\frac{3}{2}(c_4 + 4)(j - 2)(j - 3) + 4(j - 3) \tag{A6}
$$

(again ignoring the reduction due to two-point currents).

Overall, we then obtain an operation count of

$$
\sum_{j=3}^{n} (n - j + 1) \left[(3c_3 + 10)(j - 2) + \frac{3}{2}(c_4 + 4)(j - 2)(j - 3) + 4(j - 3) \right]
$$
$$
= \frac{n(n - 1)(n - 2)(n - 3)}{8} c_4 + \frac{n(n - 1)(n - 2)}{2} c_3 + \frac{(n - 1)(n - 2)(n + 3)(3n - 4)}{6} \tag{A7}
$$

In contrast to the off-shell recursion relations analyzed here, in on-shell recursion relations, the reference momenta will change as the recursion descends. It thus seems likely that the computational complexity will still be exponential for a typical color-ordered helicity amplitude. Of course, for special helicity configurations or for a moderate number of external legs, the new representations may provide a more efficient computational method.

[1] E. W. N. Glover, Nucl. Phys. Proc. Suppl. 116:3 (2003) [hep-ph/0211412].

[2] W. T. Giele and E. W. N. Glover, Phys. Rev. D46:1980 (1992); W. T. Giele, E. W. N. Glover and D. A. Kosower, Nucl. Phys. B403:633 (1993) [hep-ph/9302225]; S. Frixione, Z. Kunszt and A. Signer, Nucl. Phys. B467:399 (1996) [hep-ph/9512328]; S. Catani and M. H. Seymour, Phys. Lett. B378:287 (1996) [hep-ph/9602277]; S. Catani and M. H. Seymour, Nucl. Phys. B485:291 (1997) [Erratum-ibid. B510:503 (1998) [hep-ph/9605323]].

[3] Z. Bern, L. J. Dixon, D. C. Dunbar and D. A. Kosower, Nucl. Phys. B425:217 (1994) [hep-ph/9403226]; Z. Bern, L. J. Dixon, D. C. Dunbar and D. A. Kosower, Nucl. Phys. B435:59 (1995) [hep-ph/9409265]; Z. Bern, L. J. Dixon and D. A. Kosower, Ann. Rev. Nucl. Part. Sci. 46:109 (1996) [hep-ph/9602280];

Z. Bern, L. J. Dixon and D. A. Kosower, Nucl. Phys. Proc. Suppl. 51C:243 (1996) [hep-ph/9606378];

Z. Bern, L. J. Dixon and D. A. Kosower, JHEP 0001:027 (2000) [hep-ph/0001001].

[4] Z. Bern and A. G. Morgan, Nucl. Phys. B467:479 (1996) [hep-ph/9511336].

[5] Z. Bern, L. J. Dixon and D. A. Kosower, Nucl. Phys. B513:3 (1998) [hep-ph/9708239]; Z. Bern, L. J. Dixon and D. A. Kosower, JHEP 0408:012 (2004) [hep-ph/0404293].

[6] Z. Bern, L. J. Dixon and D. A. Kosower, JHEP 0001:027 (2000) [hep-ph/0001001]; Z. Bern, L. J. Dixon and D. A. Kosower, JHEP 0408:012 (2004) [hep-ph/0404293].

[7] D. A. Kosower and P. Uwer, Nucl. Phys. B563: 477 (1999) [hep-ph/9903515]; Z. Bern, A. De Freitas and L. J. Dixon, JHEP 0203:018 (2002) [hep-ph/0201161]; A. Brandhuber, S. McNamara, B. Spence and G. Travaglini, JHEP 0510:011 (2005) [hep-th/0506068].

[8] Z. Bern, L. J. Dixon, D. C. Dunbar and D. A. Kosower, Phys. Lett. B394:105 (1997) [hep-th/9611127].

[9] S. D. Badger, E. W. N. Glover, V. V. Khoze and P. Svrček, JHEP 0507:025 (2005) [hep-th/0504159].

[10] R. Britto, F. Cachazo and B. Feng, Nucl. Phys. B715:499 (2005) [hep-th/0412308].

[11] R. Britto, F. Cachazo, B. Feng and E. Witten, Phys. Rev. Lett. 94:181602 (2005) [hep-th/0501052].

[12] M. Luo and C. Wen, JHEP 0503:004 (2005) [hep-th/0501121]; Phys. Rev. D71:091501 (2005) [hep-th/0502009].

[13] R. Britto, B. Feng, R. Roiban, M. Spradlin and A. Volovich, Phys. Rev. D71:105017 (2005) [hep-th/0503198].

[14] J. Bedford, A. Brandhuber, B. J. Spence and G. Travaglini, Nucl. Phys. B721:98 (2005) [hep-th/0502146].

[15] F. Cachazo and P. Svrcek, hep-th/0502160.

[16] S. D. Badger, E. W. N. Glover and V. V. Khoze, JHEP 0601:066 (2006) [hep-th/0507161].

[17] Z. Bern, L. J. Dixon and D. A. Kosower, Phys. Rev. D71:105013 (2005) [hep-th/0501240]; Z. Bern, L. J. Dixon and D. A. Kosower, Phys. Rev. D72:125003 (2005) [hep-ph/0505055].

[18] Z. Bern, N. E. J. Bjerrum-Bohr, D. C. Dunbar and H. Ita, JHEP 0511:027 (2005) [hep-ph/0507019].

[19] R. Britto, F. Cachazo and B. Feng, Phys. Rev. D71:025012 (2005) [hep-th/0410179].

[20] R. Britto, F. Cachazo and B. Feng, Nucl. Phys. B725:275 (2005) [hep-th/0412103].

[21] R. Britto, F. Cachazo and B. Feng, Phys. Lett. B611:167 (2005) [hep-th/0411107].

[22] R. Roiban, M. Spradlin and A. Volovich, Phys. Rev. Lett. 94:102002 (2005) [hep-th/0412265].

[23] Z. Bern, V. Del Duca, L. J. Dixon and D. A. Kosower, Phys. Rev. D71:045006 (2005) [hep-th/0410224].

[24] Z. Bern, L. J. Dixon and D. A. Kosower, Phys. Rev. D72:045014 (2005) [hep-th/0412210].

[25] E. Witten, Commun. Math. Phys. 252:189 (2004) [hep-th/0312171].

[26] V. P. Nair, Phys. Lett. B214:215 (1988).

[27] F. A. Berends and W. T. Giele, Nucl. Phys. B306:759 (1988).

[28] D. A. Kosower, Nucl. Phys. B335:23 (1990).

[29] L. J. Dixon, in *QCD & Beyond: Proceedings of TASI '95*, ed. D. E. Soper (World Scientific, 1996) [hep-ph/9601359].

[30] Z. Bern, L. J. Dixon and D. A. Kosower, Phys. Rev. D73:065013 (2006) [hep-ph/0507005].

[31] M. L. Mangano and S. J. Parke, Phys. Rept. 200:301 (1991).

[32] P. De Causmaecker, R. Gastmans, W. Troost and T. T. Wu, Phys. Lett. B **105**, 215 (1981). F. A. Berends, R. Kleiss, P. De Causmaecker, R. Gastmans, W. Troost and T. T. Wu, Nucl. Phys. B206:61 (1982);
P. De Causmaecker, R. Gastmans, W. Troost and T. T. Wu, Nucl. Phys. B206:53 (1982);
Z. Xu, D.-H. Zhang, L. Chang, Tsinghua University preprint TUTP–84/3 (1984), unpublished;
J. F. Gunion and Z. Kunszt, Phys. Lett. B161:333 (1985). Z. Xu, D.-H. Zhang, and L. Chang, Nucl. Phys. B291:392 (1987).

[33] Z. Bern, D. Forde, D. A. Kosower and P. Mastrolia, Phys. Rev. D72:025006 (2005) [hep-ph/0412167].